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On some geometric properties of quasi-sum production models

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ABSTRACT

A production function f is called quasi-sum if there are continuous strict monotone functions $F, h_1, \dots, h_n$ with $F > 0$ such that $f(\mathbf{x}) = F(h_1(x_1) + \cdots + h_n(x_n))$ (cf. Aczél and Maksa (1996) [1]). A quasi-sum production function is called quasi-linear if at most one of $F, h_1, \dots, h_n$ is a nonlinear function. For a production function f, the graph of f is called the production hypersurface of f. In this paper, we obtain a very simple necessary and sufficient condition for a quasi-sum production function f to be quasi-linear in terms of graph of f. Moreover, we completely classify quasi-sum production functions whose production hypersurfaces have vanishing Gauss–Kronecker curvature.

1. Introduction

In microeconomics, a *production function* is a positive non-constant function that specifies the output of a firm, an industry, or an entire economy for all combinations of inputs. Almost all economic theories presuppose a production function, either on the firm level or the aggregate level. In this sense, the production function is one of the key concepts of mainstream neoclassical theories. By assuming that the maximum output technologically possible from a given set of inputs is achieved, economists using a production function in analysis are abstracting from the engineering and managerial problems inherently associated with a particular production process.

A production function is called *quasi-sum* if there are continuous strict monotone functions $h_i : \mathbb{R}_+ \to \mathbb{R}$, $i = 1, \dots, n$, and there exist an interval $I \subset \mathbb{R}$ of positive length and a continuous strict monotone function $F : I \to \mathbb{R}_+$ such that for each $\mathbf{x} \in \mathbb{R}^n_+$ we have $h_1(x_1) + \cdots + h_n(x_n) \in I$ and

$$f(\mathbf{x}) = F(h_1(x_1) + \cdots + h_n(x_n)). \tag{1.1}$$

The justification for studying production functions of quasi-sum form is that these functions appear as solutions of the general bisymmetry equation and they are related to the problem of consistent aggregation (cf. Aczél and Maksa [1]). The class of quasi-sum production functions includes the well-known generalized Cobb–Douglas production functions and the ACMS production functions (cf. Section 2). A quasi-sum production function is called *quasi-linear* if at most one of $F, h_1, \dots, h_n$ in (1.1) is a nonlinear function.

Each production function f can be identified with the graph of f, i.e., the non-parametric hypersurface of the Euclidean $(n+1)$-space $\mathbb{E}^{n+1}$ defined by

$$L(x_1, \dots, x_n) = (x_1, \dots, x_n, f(\mathbf{x})). \tag{1.2}$$

The graph of f is known as the *production hypersurface* of f (cf. [2–4]).

In this paper we study quasi-sum production functions via their production hypersurfaces. As results, we obtain a very simple characterization of quasi-linear production functions. Moreover, we completely classify quasi-sum production functions whose production hypersurfaces have vanishing Gauss–Kronecker curvature.

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2. CD and ACMS production functions

Cobb and Douglas [5] introduced a famous two-factor production function:

$$Y = bL^kC^{1-k}, \tag{2.1}$$

where L represents the labor input, C is the capital input, b is the total factor productivity and Y is the total production. In its generalized form the Cobb–Douglas production function may be written as

$$Q(\mathbf{x}) = bx_1^{\alpha_1}\cdots x_n^{\alpha_n}, \quad \mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n_+, \tag{2.2}$$

where b is a positive constant and $\alpha_1, \ldots, \alpha_n$ are some nonzero constants.

Arrow et al. [6] introduced another two-factor production function given by

$$Q = F \cdot (aK^r + (1-a)L^r)^{\frac{1}{r}}, \tag{2.3}$$

where Q is the output, F the factor productivity, a the share parameter, K, L the primary production factors, $r = (s-1)/s$, and $s = 1/(1-r)$ is the elasticity of substitution. The generalized form of ACMS production function is given by

$$Q(\mathbf{x}) = b\left(\sum_{i=1}^n a_i^\rho x_i^\rho\right)^{\frac{h}{\rho}}, \quad \mathbf{x} = (x_1, \ldots, x_n) \in \mathbf{R}^n_+, \tag{2.4}$$

where a_i, b, h, ρ are constants with $b, h > 0$, $\rho < 1$ and $a_i, \rho \neq 0$.

Some geometric properties of CD- and ACMS-production hypersurfaces have been studied recently in [2–4].

The most common quantitative indices of production factor substitutability are forms of the elasticity of substitution. The elasticity of substitution was originally introduced by Hicks [7] in case of two inputs for the purpose of analyzing changes in the income shares of labor and capital. Elasticity of substitution is the elasticity of the ratio of two inputs to a production function with respect to the ratio of their marginal products. It tells how easy it is to substitute one input for the other.

Allen and Hicks [8] suggested a generalization of Hicks' original two variable elasticity. Let f be a twice differentiable production function with non-vanishing first partial derivatives. Put

$$H_{ij}(\mathbf{x}) = \frac{\frac{1}{x_i f_{x_i}} + \frac{1}{x_j f_{x_j}}}{-\frac{f_{x_i x_i}}{f_{x_i}^2} + \frac{2f_{x_i x_j}}{f_{x_i} f_{x_j}} - \frac{f_{x_j x_j}}{f_{x_j}^2}}, \quad 1 \le i \neq j \le n, \tag{2.5}$$

for $\mathbf{x} \in \mathbb{R}^n_+$, where $f_{x_i} = \frac{\partial f}{\partial x_i}$, $f_{x_i x_j} = \frac{\partial^2 f}{\partial x_i \partial x_j}$, all partial derivatives are taken at the point $\mathbf{x}$ and the denominator is assumed to be different from zero. The H_{ij} is called the *Hicks elasticity of substitution* of the i-th production variable with respect to the j-th production variable.

A twice differentiable production function f with nonzero first partial derivatives is said to satisfy the CES (constant elasticity of substitution) property if there is a nonzero constant $\sigma \in \mathbb{R}$ such that

$$H_{ij}(\mathbf{x}) = \sigma \quad \text{for } \mathbf{x} \in \mathbb{R}^n_+ \text{ and } 1 \le i \neq j \le n. \tag{2.6}$$

It is well-known that both the generalized CD production function and the ACMS production function satisfy the CES property. It is easy to verify that $H_{ij}(\mathbf{x}) = 1$ for the generalized CD production function and $H_{ij}(\mathbf{x}) = 1/\rho$ for the ACMS production function if $\rho \neq 1$. For $\rho = 1$ the denominator of H_{ij} is zero, hence it is not defined.

A production function $f(\mathbf{x})$ is said to be h-homogeneous or *homogeneous of degree h*, if given any positive constant t,

$$f(tx_1, \ldots, tx_n) = t^h f(x_1, \ldots, x_n) \tag{2.7}$$

for some constant h. CD and ACMS production functions are homogeneous.

The author has studied geometric properties of h-homogeneous production hypersurfaces in [2]. Also, the author has completely classified h-homogeneous production functions with the CES property in the following theorem [9].

Theorem 2.1. *Let f be a twice differentiable n-factors h-homogeneous production function with non-vanishing first partial derivatives. If f satisfies the CES property, then it is either the generalized Cobb–Douglas production function or the ACMS production function.*

Remark 2.1. When $n = 2$, Theorem 2.1 is due to Losonczi [10].

3. Curvatures of production hypersurfaces

Let M be a hypersurface of a Euclidean $(n+1)$-space $\mathbb{E}^{n+1}$. The *Gauss map* $\nu : M \to S^{n+1}$ maps M to the unit hypersphere S^n of $\mathbb{E}^{n+1}$. The Gauss map is a continuous map such that $\nu(p)$ is a unit normal vector $\xi(p)$ of M at $p \in M$. The Gauss map can always be defined locally, i.e., on a small piece of the hypersurface. It can be defined globally if the hypersurface is orientable.

The differential $d\nu$ of ν can be used to define a type of extrinsic quantity, known as the *shape operator*. Since each tangent space T_pM is an inner product space, the shape operator S_p can be defined as a linear operator on T_pM by

$$g(S_p v, w) = g(d\nu(v), w) \tag{3.1}$$

for $v, w \in T_pM$, where g is the induced metric on M. The eigenvalues of the shape operator are called principal curvatures. The determinant of the shape operator S_p, denoted by $G(p)$, is called the *Gauss–Kronecker curvature*. When $n = 2$, the Gauss–Kronecker curvature is called the *Gauss curvature*.

Curves on a Riemannian manifold N which minimize length between the endpoints are called geodesics. Mathematically, they are described using partial differential equations from the calculus of variations. For a given unit tangent vector $u \in T_pN$, there exists a unique unit speed geodesic $\gamma_u(s)$ in N through p such that $\gamma_u'(0) = u$. For a given 2-plane section π of T_pN, all of geodesics through p and tangent to π form a surface in some neighborhood of p. The Gauss curvature of this surface at p is called the *sectional curvature* of $\pi \subset T_pN$.

On N there is a unique affine connection ∇, called the *Levi–Civita connection* which preserves the metric, i.e, $\nabla g = 0$, and torsion-free, i.e., $\nabla_X Y - \nabla_Y X = [X, Y]$ for vector fields X and Y on N, where $[\ ,\ ]$ is the Lie bracket.

The *Riemann curvature tensor* R is given in terms of ∇ by the following formula:

$$R(u, v)w = \nabla_u \nabla_v w - \nabla_v \nabla_u w - \nabla_{[u,v]} w. \tag{3.2}$$

A Riemannian manifold is called a *flat space* if its Riemann curvature tensor vanishes identically.

The following result is well-known (see, e.g. [2]).

Proposition 3.1. *For the production hypersurface of* $\mathbb{E}^{n+1}$ *defined by*

$$L(\mathbf{x}) = (x_1, \ldots, x_n, f(x_1, \ldots, x_n)), \tag{3.3}$$

we have:

(i) *The Gauss–Kronecker curvature G is given by*

$$G = \frac{\det(f_{x_i x_j})}{w^{n+2}} \tag{3.4}$$

with $w = \sqrt{1 + \sum_{i=1}^n f_{x_i}^2}$.

(ii) *The sectional curvature K_{ij} of the plane section spanned by* $\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}$ *is given by*

$$K_{ij} = \frac{f_{x_i x_i} f_{x_j x_j} - f_{x_i x_j}^2}{w^2(1 + f_{x_i}^2 + f_{x_j}^2)}. \tag{3.5}$$

(iii) *The Riemann curvature tensor R and the metric tensor g satisfy*

$$g\left(R\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)\frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_\ell}\right) = \frac{f_{x_i x_\ell} f_{x_j x_k} - f_{x_i x_k} f_{x_j x_\ell}}{w^4}. \tag{3.6}$$

4. Quasi-sum production models

The following theorem provides a very simple necessary and sufficient condition for a quasi-sum production function with more than two factors to be quasi-linear.

Theorem 4.1. *A twice differentiable quasi-sum production function with more than two factors is quasi-linear if and only if its production hypersurface is a flat space.*

Proof. Let f be a twice differentiable quasi-sum production function with more than two factors given by

$$f(\mathbf{x}) = F(h_1(x_1) + \cdots + h_n(x_n)), \tag{4.1}$$

where $F, h_1, \ldots, h_n$ are continuous strict monotone functions. Thus we have

$$F', h_1', \ldots, h_n' \neq 0, \tag{4.2}$$

at every point, where $h_j' = \frac{dh_j}{dx_j}$ for $j = 1, \ldots, n$.

From (4.1) we find

$$f_{x_i x_i} = F'h_i'' + F''h_i'^2, \qquad f_{x_i x_j} = F''h_i'h_j', \quad 1 \le i \ne j \le n. \tag{4.3}$$

Let us assume that the production hypersurface of the production function (4.1) is a flat space. Then (4.3) and statement (ii) of Proposition 3.1 imply that

$$\{h_i''(x_i)h_j'^2(x_j)+h_j''(x_j)h_i'^2(x_i)\}F''=-h_i''(x_i)h_j''(x_j)F',\quad 1\le i\ne j\le n. \tag{4.4}$$

Case (a): *At least one of* $h_1'',\dots,h_n''$ *vanishes.* Without loss of generality, we may assume $h_1''=0$. Then (4.2) and (4.4) imply that either $F''=0$ or

$$h_j''(x_j)=0,\quad j=2,\dots,n. \tag{4.5}$$

Case (a.1): $F''=0$. It follows from (4.4) that

$$h_i''(x_i)h_j''(x_j)=0,\quad 2\le i\ne j\le n.$$

Thus at most one of $h_2'',\dots,h_n''$ is nonzero. Without loss of generality, we may assume that $h_2''=\cdots=h_{n-1}''=0$. After combining this with (4.2) together with $F''=h_1''=0$, we obtain

$$F(u)=\alpha u+\beta, \tag{4.6}$$

$$h_i(x_i)=\alpha_i x_i+\beta_i,\quad i=1,\dots,n-1, \tag{4.7}$$

for some constants $\alpha,\beta,\alpha_i,\beta_i$ with $\alpha,\alpha_i\ne 0$. After substituting (4.6) and (4.7) into (4.1) we obtain

$$f(\mathbf{x})=a_1x_1+\cdots+a_{n-1}x_{n-1}+\varphi(x_n) \tag{4.8}$$

for some nonzero constants $a_1,\dots,a_{n-1}$ and a strict monotone function φ. Thus, f is quasi-linear.

Conversely, if the production function f is given by (4.8), then we have $f_{x_ix_j}=0$ for all $i,j\in\{1,\dots,n\}$, except $f_{x_nx_n}$. Therefore, it follows from statement (iii) of Proposition 3.1 that the production hypersurface is a flat space.

Case (a.2): $F''\ne 0$. In this case we have (4.5) as well as $h_1''=0$. Thus

$$h_i(x_i)=a_ix_i+b_i,\quad i=1,\dots,n, \tag{4.9}$$

for some constants $a_i\ne 0$, b_i. By substituting (4.9) into (4.1), we obtain

$$f(\mathbf{x})=F(a_1x_1+\cdots+a_nx_n+c) \tag{4.10}$$

with $c=b_1+\cdots+b_n$. Thus the production is quasi-linear.

Conversely, if the production function f is given by (4.10), then we have

$$f_{x_ix_j}=a_ia_jF'',\quad i,j=1,\dots,n. \tag{4.11}$$

Therefore, the production hypersurface is flat according to (3.6) and (4.11).

Case (b): $h_1'',\dots,h_n''$ are nonzero. In this case, it follows from (4.4) that F'' is also nonzero. Thus, (4.4) can be rewritten as

$$\frac{h_i'^2(x_i)}{h_i''(x_i)}+\frac{h_j'^2(x_j)}{h_j''(x_j)}=-\frac{F'}{F''},\quad 1\le i\ne j\le n. \tag{4.12}$$

Since $n\ge 3$, (4.12) implies that

$$\frac{h_i'^2(x_i)}{h_i''(x_i)}=\frac{h_j'^2(x_j)}{h_j''(x_j)}$$

for $i,j=1,\dots,n$. Hence we derive from (4.12) that

$$h_i''(x_i)-bh_i'^2(x_i)=0,\quad i=1,\dots,n, \tag{4.13}$$

$$2F''(u)+bF'(u)=0, \tag{4.14}$$

for some nonzero constant b. After solving (4.13) and (4.14) we get

$$h_i(x_i)=\frac{1}{b}\ln\left(\frac{\beta_i}{x_i-k_i}\right),\quad i=1,\dots,n, \tag{4.15}$$

$$F(u)=c_1e^{-\frac{b}{2}u}+c_2, \tag{4.16}$$

for some constants β_i,c_1,c_2,k_i. Substituting (4.15) and (4.16) into (4.1) yields

$$f(\mathbf{x})=c_0\sqrt{(x_1-k_1)\cdots(x_n-k_n)}, \tag{4.17}$$

where c_0 is a nonzero number. It follows from (4.17) that

$$f_{x_ix_i} = \frac{-c_0\sqrt{(x_1-k_1)\cdots(x_n-k_n)}}{4(x_i-k_i)^2},$$
$$f_{x_ix_j} = \frac{c_0\sqrt{(x_1-k_1)\cdots(x_n-k_n)}}{4(x_i-k_i)(x_j-k_j)}, \tag{4.18}$$

for $1 \le i \ne j \le n$. From (4.18) we conclude that the Riemann curvature tensor of the production hypersurface satisfies

$$g\left(R\left(\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_3}\right)\frac{\partial}{\partial x_3},\frac{\partial}{\partial x_2}\right) = \frac{f_{x_1x_2}f_{x_3x_3}-f_{x_1x_3}f_{x_2x_3}}{w^4}$$
$$= -\frac{c_0(x_4-k_4)\cdots(x_n-k_n)}{8(x_3-k_3)w^4} \ne 0. \tag{4.19}$$

Thus the production hypersurface is a non-flat space, which is a contradiction. □

A Riemannian space is called Ricci-flat if its Ricci tensor vanishes (cf. e.g. [11]). Since Ricci-flat 3-manifolds are flat spaces, Theorem 4.1 implies the following.

Theorem 4.2. *A three-factor quasi-sum production function is quasi-linear if and only if its production hypersurface is a Ricci-flat space.*

5. Quasi-sum production hypersurfaces satisfying $G = 0$

The next result completely classifies quasi-sum production functions whose production hypersurfaces have null Gauss–Kronecker curvature.

Theorem 5.1. *Let f be a twice differentiable quasi-sum production function. Then the production hypersurface of f has vanishing Gauss–Kronecker curvature if and only if, up to translations, f is one of the following:*

(a) $f = ax_1 + \sum_{i=2}^n \varphi_i(x_i)$, *where a is a nonzero constant and $\varphi_2,\ldots,\varphi_n$ are strict monotone functions;*
(b) $f = F\big(a_1x_1 + a_2x_2 + \sum_{i=3}^n \varphi_i(x_i)\big)$, *where a_1, a_2 are nonzero constants and $F, \varphi_3,\ldots,\varphi_n$ are strict monotone functions;*
(c) *f is a generalized Cobb–Douglas function given by $f = \gamma x_1^{\alpha_1}\cdots x_n^{\alpha_n}$ for some nonzero constants $\gamma, \alpha_1,\ldots,\alpha_n$ satisfying $\sum_{i=1}^n \alpha_n = 1$;*
(d) $f = \left(\sum_{i=1}^n a_i x_i^{\frac{\varepsilon-1}{\varepsilon-2}}\right)^{\frac{\varepsilon-2}{\varepsilon-1}}$, *where a_i, ε are constants with $a_i \ne 0$ and with $\varepsilon \ne 1, 2$;*
(e) $f = a\ln\left(\sum_{i=1}^n b_i e^{r_ix_i}\right)$ *for some nonzero constants a, b_i, r_i.*

Proof. Let f be a twice differentiable quasi-sum production function given by

$$f(\mathbf{x}) = F(h_1(x_1)+\cdots+h_n(x_n)), \tag{5.1}$$

where $F, h_1,\ldots,h_n$ are continuous strict monotone functions. Thus we have

$$f_{x_ix_i} = F'h_i'' + F''h_i'^2, \qquad f_{x_ix_j} = F''h_i'h_j', \tag{5.2}$$

for $1 \le i \ne j \le n$. Therefore, after applying statement (i) of Proposition 3.1, we conclude that the Gauss–Kronecker curvature of f is given by

$$G(\mathbf{x}) = \frac{F'(u)^{n-1}}{w^{n+2}}\left\{F'(u)\prod_{i=1}^n h_i''(x_i) + F''(u)\sum_{j=1}^n h_1''(x_1)\cdots h_{j-1}''(x_{j-1})h_j'(x_j)^2h_{j+1}''(x_{j+1})\cdots h_n''(x_n)\right\}.$$

Therefore the production hypersurface has vanishing Gauss–Kronecker curvature if and only if we have

$$F''(u)\sum_{j=1}^n h_1''(x_1)\cdots h_{j-1}''(x_{j-1})h_j'(x_j)^2h_{j+1}''(x_{j+1})\cdots h_n''(x_n) + F'(u)\prod_{i=1}^n h_i''(x_i) = 0. \tag{5.3}$$

Case (i): $F'' = 0$. Since $F' \ne 0$, (5.3) implies that $\prod_{i=1}^n h_i''(x_i) = 0$. Without loss of generality, we may assume $h_1'' = 0$. Thus we have

$$F(u) = \alpha u + \beta, \qquad h_1(x_1) = a_1x_1 + b_1, \tag{5.4}$$

for some constants α, β, a_1, b_1 with $\alpha, a_1 \ne 0$. After substituting (5.4) into (5.1) we obtain

$$f(\mathbf{x}) = ax_1 + \varphi_2(x_2) + \cdots + \varphi_n(x_n) \tag{5.5}$$

for some function $\varphi_2,\ldots,\varphi_n$ and nonzero constant a. Therefore we get case (a) of the theorem.

Case (ii): *$F'' \neq 0$ and at least one of $h_1'', \dots, h_n''$ vanishes.* Without loss of generality, we may assume that $h_1'' = 0$. Hence after applying (5.3) and the fact that h_1 is strictly monotone we get

$$h_2''(x_2) \cdots h_n''(x_n) = 0. \tag{5.6}$$

Without loss of generality we may assume from (5.6) that $h_2'' = 0$. Thus we have

$$h_1(x_1) = a_1x_1 + b_1, \qquad h_2(x_2) = a_2x_2 + b_2, \tag{5.7}$$

for some constants a_1, a_2, b_1, b_2 with $a_1, a_2 \neq 0$. Therefore the production function takes the form:

$$f(\mathbf{x}) = F(a_1x_1 + a_2x_2 + \varphi_3(x_3) + \cdots + \varphi_n(x_n)) \tag{5.8}$$

for some nonzero constants a_1, a_2 and strict monotone functions $\varphi_3(x_3), \dots, \varphi_n(x_n)$. This gives case (b) of the theorem.

Case (iii): *$F'', h_1'', \dots, h_n''$ are nonzero.* In this case, Eq. (5.3) can be expressed as

$$\frac{h_1'^2(x_1)}{h_1''(x_1)} + \cdots + \frac{h_n'^2(x_n)}{h_2''(x_n)} + \frac{F'(u)}{F''(u)} = 0, \quad u = \sum_{i=1}^n h_i(x_i). \tag{5.9}$$

By taking the partial derivative of (5.9) with respect to x_i, we obtain

$$\frac{h_i'(x_i)h_i'''(x_i)}{h_i''(x_i)^2} = 3 - \frac{F'(u)F'''(u)}{F''(u)^2}. \tag{5.10}$$

Therefore, after taking the partial derivative of (5.10) with respect to x_j with $j \neq i$, we derive that

$$F'''(u)\{F''(u)^2 - 2F'(u)F'''(u)\} + F'(u)F''(u)F^{(iv)}(u) = 0. \tag{5.11}$$

We divide the proof of case (iii) into several cases.

Case (iii.a): $F''' = 0$. From (5.10) we find

$$h_i'(x)h_i'''(x_i) = 3h_i''(x_i)^2, \quad i = 1, \dots, n. \tag{5.12}$$

After solving (5.12) we find

$$h_i(x_i) = a_i\sqrt{x_i + r_i} + b_i \tag{5.13}$$

for some nonzero constants a_i and constants r_i, b_i.

Also, from $F''' = 0$ we get

$$F(u) = \gamma u^2 + c_1u + c_2 \tag{5.14}$$

for some constants γ, c_1, c_2. Now, after substituting (5.13) and (5.14) into (5.1) we obtain

$$f = \gamma\left(\sum_{i=1}^n a_i\sqrt{x_i + r_i} + b\right)^2 + c_1\left(\sum_{i=1}^n a_i\sqrt{x_i + r_i} + b\right) + c_2, \tag{5.15}$$

with $b = \sum_{i=1}^n b_i$.

It is straight-forward to verify that the production hypersurface of f given by (5.15) satisfies condition (5.3) if and only if $b = c_1 = 0$. Therefore, after applying a suitable translation, we obtain case (d) of the theorem with $\varepsilon = 0$.

Case (iii.b): $F''' \neq 0$. In this case, equation (5.11) yields

$$\frac{F''(u)}{F'(u)} + \frac{F^{(iv)}(u)}{F'''(u)} = \frac{2F'''(u)}{F''(u)},$$

which implies that

$$F'(u)F'''(u) = \varepsilon F''(u)^2 \tag{5.16}$$

for some nonzero constant ε.

Now, we divide the proof of case (iii.b) into four cases based on the value of ε.

Case (iii.b.1): $\varepsilon = 1$. After solving (5.16) we get

$$F(u) = \mu e^{bu} + r \tag{5.17}$$

for some constant b, r, μ with $\mu, b \neq 0$. By substituting (5.17) into (5.10), we get

$$h_i'(x_i)h_i'''(x_i) = 2h_i''(x_i)^2, \quad i = 1, \ldots, n. \tag{5.18}$$

By solving (5.18) we obtain

$$h_i(x) = a_i \ln(x_i + b_i) + \mu_i \tag{5.19}$$

for some constants a_i, b_i, μ_i with $a_i \neq 0$. Therefore, from (5.1), (5.17), (5.19) and $u = h_1(x_1) + \cdots + h_n(x_n)$, we conclude that f takes the form:

$$f = a(x_1 + b_1)^{ba_1} \cdots (x_n + b_n)^{ba_n} + r \tag{5.20}$$

for some constants a, a_i, b_i, r with $a, a_i \neq 0$. Now, by applying (5.20) and statement (i) of Proposition 3.1 we conclude that the Gauss–Kronecker curvature vanishes if and only if $b\sum_{i=1}^n a_i = 1$ holds. Therefore, after applying suitable translation we obtain case (c) of the theorem.

Case (iii.b.2): $\varepsilon = 2$. After solving (5.16) we get

$$F(u) = a\ln(bu + c) \tag{5.21}$$

for some constant a, b, c with $a, b \neq 0$. By substituting (5.21) into (5.10), we get

$$h_i'(x_i)h_i'''(x_i) = h_i''(x_i)^2, \quad i = 1, \ldots, n. \tag{5.22}$$

By solving (5.22) we obtain

$$h_i(x_i) = b_i e^{r_i x_i} + \mu_i \tag{5.23}$$

for some constants b_i, r_i, μ_i with $b_i, r_i \neq 0$. Therefore it follows from (5.1), (5.21), (5.23) that f takes the form:

$$f = a\ln\left(\sum_{i=1}^n b_i e^{r_i x_i} + k\right) \tag{5.24}$$

for some constants a, b_i, k, r_i with $a, b_i, r_i \neq 0$. By applying (5.24) and statement (i) of Proposition 3.1 we conclude that the Gauss–Kronecker curvature of the production hypersurface vanishes if and only if $k = 0$. Consequently, after a suitable translation, we obtain case (e) of the theorem.

Case (iii.b.3): $\varepsilon = 3$. After solving (5.16) we get

$$F(u) = a\sqrt{u + b} + c \tag{5.25}$$

for some constant a, b, c with $a \neq 0$. By substituting (5.25) into (5.10), we obtain $h_1'''(x_1) = \cdots = h_n'''(x_n) = 0$. Thus

$$h_i(x_i) = r_i x_i^2 + s_i x_i + t_i \tag{5.26}$$

for some constants r_i, s_i, t_i. From (5.1), (5.25) and (5.26), we conclude that f takes the form:

$$f = \left(\sum_{i=1}^n b_i(x_i + c_i)^2 + d\right)^{\frac{1}{2}} + c \tag{5.27}$$

for some constants b_i, c_i, c, d with $b_i \neq 0$. Now, it is direct to verify that the production hypersurface associated with the production function given by (5.27) has vanishing Gauss–Kronecker curvature vanishes identically if and only if $d = 0$. Consequently, after a suitable translation, we obtain case (d) with $\varepsilon = 3$.

Case (iii.b.4): $\varepsilon \neq 1, 2, 3$. After solving (5.16) we get

$$F(u) = a(u + b)^{\frac{\varepsilon-2}{\varepsilon-1}} + c \tag{5.28}$$

for some constant a, b, d with $a \neq 0$. By substituting (5.28) into (5.10), we get

$$h_i'(x_i)h_i'''(x_i) = (3 - \varepsilon)h_i''(x_i)^2. \tag{5.29}$$

After solving (5.29) we obtain

$$h_i(x_i) = \delta_i((\varepsilon - 2)x_i + \beta_i)^{\frac{\varepsilon-1}{\varepsilon-2}} + \gamma_i, \quad i = 1, \ldots, n, \tag{5.30}$$

for some constants $\beta_i, \gamma_i, \delta_i$. From (5.1), (5.28), (5.30) and $u = \sum_{i=1}^{n} h_i(x_i)$, we conclude that f takes the form:

$$f = \left(\sum_{i=1}^{n} a_i (x + b_i)^{\frac{\varepsilon-1}{\varepsilon-2}} + \gamma \right)^{\frac{\varepsilon-2}{\varepsilon-1}} + c \tag{5.31}$$

for some constants a_i, b_i, γ, c with $a_i \neq 0$. It is direct to verify that the production hypersurface of (5.31) has vanishing Gauss–Kronecker curvature if and only if $\gamma = 0$. Therefore, after a suitable translation we obtain case (d) of the theorem.

Conversely, it is direct to verify all of the production hypersurfaces defined by the functions in cases (a)-(e) have vanishing Gauss–Kronecker curvature. $\square$

Theorem 5.1 implies the following result for quasi-sum production functions with two-factors.

Corollary 5.1. *Let $f(x, y)$ be a twice differentiable quasi-sum production function. Then the production surface of f is a flat surface if and only if, up to translations, f is one of the following:*

(a) *a quasi-linear production function;*
(b) *f is a Cobb–Douglas function, i.e. $f = ax^r y^{1-r}$ for some nonzero constants a, r with $r \neq 1$;*
(c) *f is an ACMS function given by $f = \left(ax^{\frac{\varepsilon-1}{\varepsilon-2}} + by^{\frac{\varepsilon-1}{\varepsilon-2}}\right)^{\frac{\varepsilon-2}{\varepsilon-1}}$ with $\varepsilon \neq 1, 2$;*
(d) *$f = a \ln(be^{rx} + ce^{sy})$ for some nonzero constants a, b, c, r, s.*

Remark 5.1. Theorem 4.1 together with Corollary 5.1 completely classify all quasi-sum production functions whose production hypersurfaces are flat spaces.

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